10:00 - 11:00
32-141

MIT ID# (last four digits) SOLUTIONS

Unified Quiz M1

March 5, 2008

M - PORTION

- Put the last four digits of your MIT ID # on each page of the exam.
- Read all questions carefully.
- Do all work on that question on the page(s) provided. Use back of the page(s) if necessary.
- Show all your work, especially intermediate results. Partial credit cannot be given without intermediate results.
- Show the logical path of your work. Explain clearly your reasoning and what you are doing. *In some cases, the reasoning is worth as much (or more) than the final answers.*
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Be sure to show the appropriate units throughout. Answers are not correct without the units.
- Report significant digits only.
- Box your final answers.
- ***Calculators are allowed.***
- ***Print-outs of Handouts "HO-M-8" and "HO-M-11" along with 2 sides of pages of handwritten material are allowed.***

EXAM SCORING

#1M (50%)	
#2M (50%)	
FINAL SCORE	

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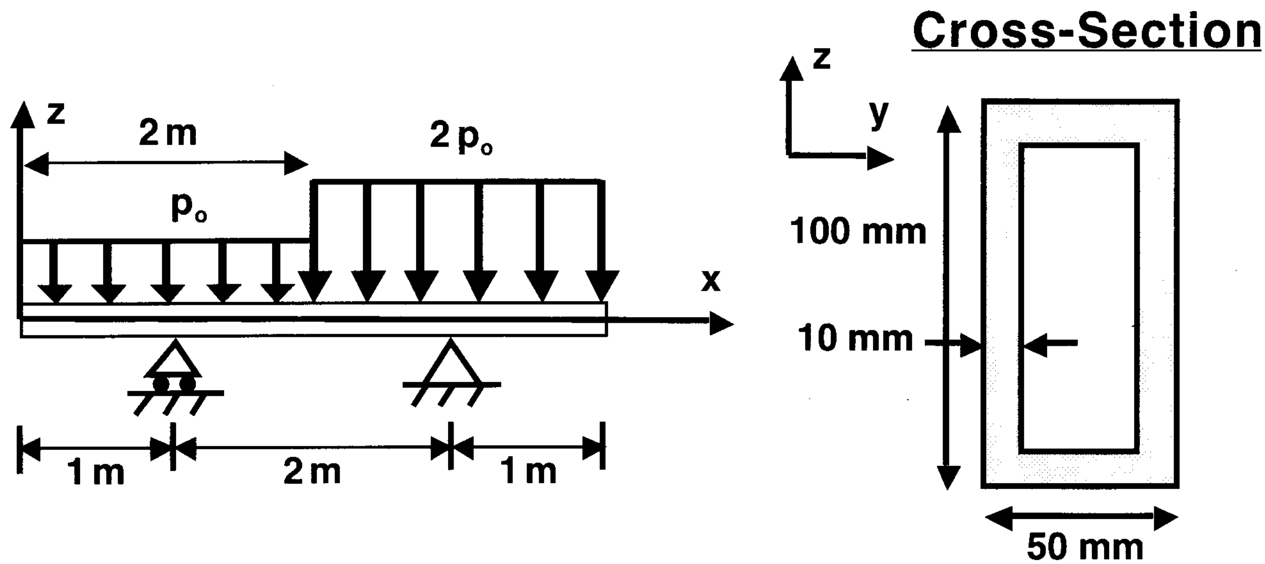
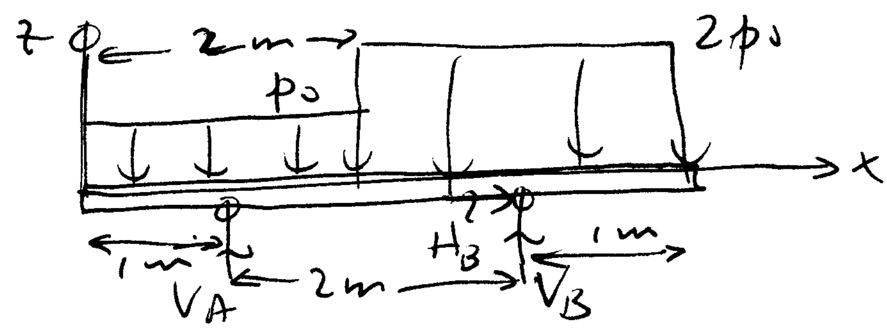
PROBLEM #1M (50%)

A 4-meter long steel beam ($E = 200$ GPa, $\nu = 0.3$) is supported by a roller-pin configuration with a roller inboard 1 meter from one end and a pin 1 meter inboard from the other. The beam has a rectangular box cross-section with outer dimensions of 100 mm by 50 mm and a wall thickness of 10 mm. The beam is loaded by two constant load segments: p_0 in intensity for the first half and double that for the second half. This overall configuration is shown in the accompanying figure.



(a) Sketch the shear force and bending moment resultant distributions as a function of position along the beam. Be sure to note the key values of each and their locations.

Draw the Free Body Diagram:



Apply equilibrium:

$$\Sigma F_x = 0 \xrightarrow{+} \Rightarrow H_B = 0$$

$$\Sigma F_z = 0 \uparrow + \Rightarrow V_A + V_B - p_0(2m) - 2p_0(2m) = 0$$

$$\Rightarrow V_A + V_B = 6p_0(1m)$$

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PROBLEM #1M (continued)

$$\Sigma M_o = 0 \; (+ \Rightarrow V_A(1m) + V_B(3m) - \int_0^{2m} p_o x\,dx - \int_{2m}^{4m} 2p_o x\,dx = 0$$

$$\Rightarrow V_A(1m) + V_B(3m) = p_o \frac{x^2}{2}\Big]_0^{2m} + 2p_o \frac{x^2}{2}\Big]_{2m}^{4m}$$

$$= 2p_o(1m)^2 + 12p_o(1m)^2 = 14p_o(1m)^2$$

From the $\Sigma F_z = 0$ equation:

$$V_A = 6p_o(1m) - V_B$$

Using in above:

$$2V_B + 6p_o(1m) = 14p_o(1m) \Rightarrow V_B = 4p_o(1m)$$
$$V_A = 2p_o(1m)$$

→ To get the shear diagram use:

$$\frac{dS}{dx} = q(x)$$

And we can note that for shear:

- $S = 0$ @ $x = 0$ (no tip load)
- slope of $-p_o$ $\quad 0 < x < 2m$
- value of $-p_o(1m)$ @ $x = 1m$ and jump by $+V_A = 2p_o(1m)$
- slope change @ $x = 2m$ to $-2p_o$ and maintained for $2m < x < 4m$
  ↳ value of 0 there
- value of $-p_o(2m)$ @ $x = 3m$
- jump of $+V_B = 4p_o(1m)$ @ $x = 3m$
- $S = 0$ @ $x = 4m$ (no tip load)

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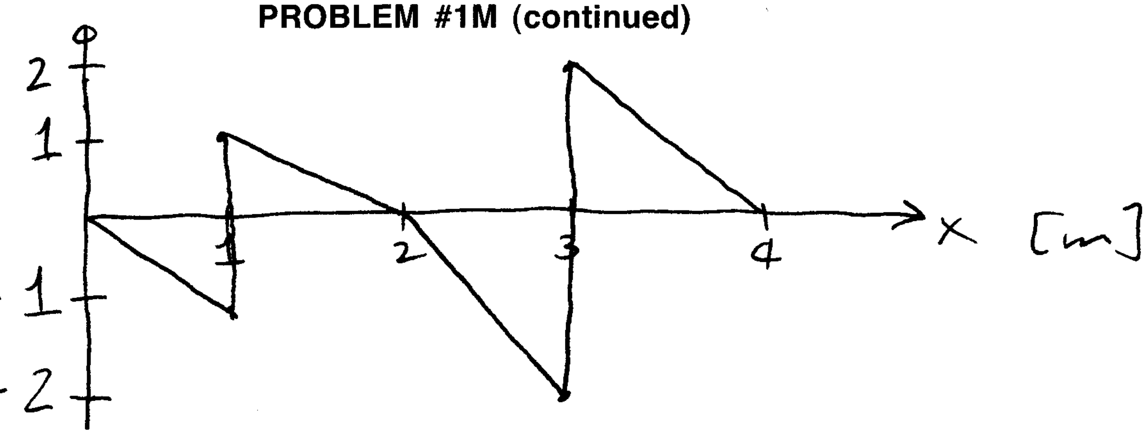
PROBLEM #1M (continued)

Thus:

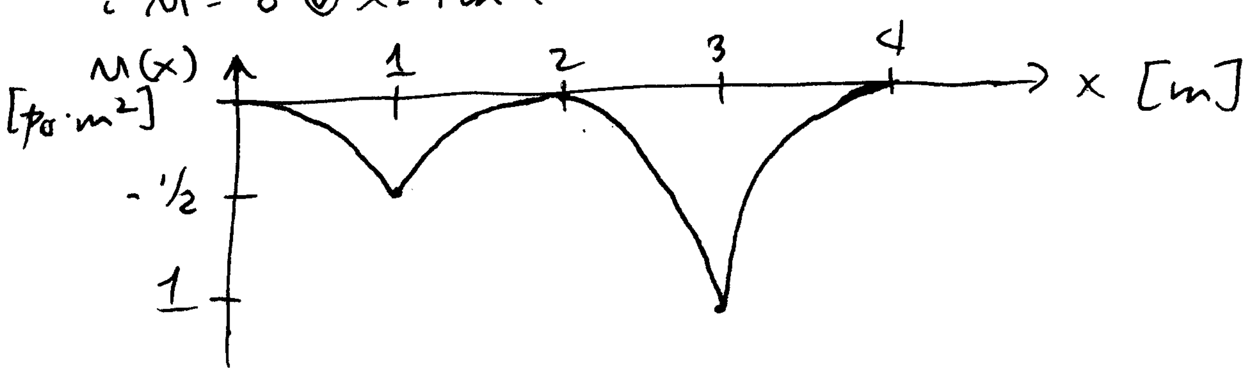


→ To get the moment diagram use:

$$\frac{dM}{dx} = S(x)$$

And we can note that:

- $M = 0$ @ $x = 0$ (free end)
- negative slope before at 0 @ $x = 0$ and increases to $x = 1m$
- $M = -\frac{p_0}{2} \cdot m^2$ @ $x = 1m$ (via cut and equilibrium)
- change in slope to positive @ $x = 1m$ and decreases to 0 @ $x = 2m$ (antisymmetric)
- $M = 0$ @ $x = 2m$ (via cut and equilibrium)
- change in slope to negative @ $x = 2m$ and increases to $x = 3m$
- change in slope to positive @ $x = 3m$ and decreases to 0 @ $x = 4m$ (antisymmetric)
- $M = -p_0 \cdot m^2$ @ $x = 3m$ (via cut and equilibrium)
- $M = 0$ @ $x = 4m$ (free end)



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PROBLEM #1M (continued)

(b) Determine the x-location of the maximum axial stress (i.e. σ_{xx}).

The applicable equation is:

$$\sigma_{xx} = -\frac{Mz}{I}$$

The maximum value of $\frac{z}{I}$ will not change with x so the maximum value of σ_{xx} occurs where the magnitude of $M(x)$ is a maximum:

$$\boxed{x = 3m}$$

(c) Determine the x-location of the maximum shear stress (i.e. σ_{xz}).

The applicable equation is:

$$\sigma_{xz} = -\frac{SQ}{Ib}$$

The maximum value of $\frac{Q}{Ib}$ will not change with x, so the maximum value of σ_{xz} occurs where the magnitude of $S(x)$ is a maximum:

$$\boxed{x = 3m}$$

PROBLEM #1M (continued)

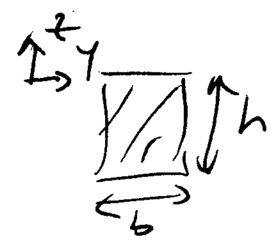
(d) You need to reduce the maximum deflection by a factor of two and can do so by changing the wall thickness while keeping the outer box dimensions constant. Clearly indicate and explain the analysis needed to determine the new thickness. You can use and set up equations, but **do not** solve any resulting final equations.

The Moment Curvature Relation is

$$\frac{d^2w}{dx^2} = \frac{M}{EI}$$

So with all else constant to reduce the deflection by a factor of 2, one must increase I by a factor of 2.

→ Initial I: for a rectangle solid $I = \frac{bh^3}{12}$

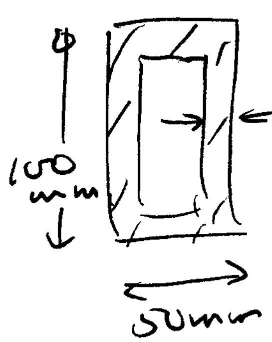


For concentrically positioned rectangles:

$$I_{box} = I_{outer} - I_{inner}$$

Here: outer box $h = 100\text{ mm}$, $b = 50\text{ mm}$ $\Rightarrow I_{outer} = \frac{(50\text{ mm})(100\text{ mm})^3}{12}$

$$= \frac{5000 \times 10^4}{12}\text{ mm}^4$$



inner box $h = 100\text{ mm} - 2(10\text{ mm}) = 80\text{ mm}$

$b = 50\text{ mm} - 2(10\text{ mm}) = 30\text{ mm}$

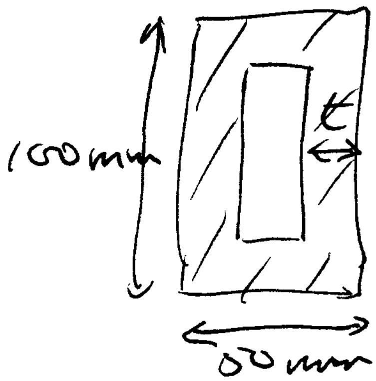
$$\Rightarrow I_{inner} = \frac{(30\text{ mm})(80\text{ mm})^3}{12}$$

$$= \frac{1536 \times 10^4}{12}\text{ mm}^4$$

$$I_{box_{original}} = \frac{1}{2}(5000 - 1531) \times 10^4\text{ mm}^4 = 2.886 \times 10^7\text{ mm}^4$$

PROBLEM #1M (continued)

The new cross-section needs to have twice this I: $I_{box_{new}} = 5.772\times10^{7}\ mm^4$



The outer dimensions are the same, but the inner dimensions change as the wall thickness changes

$h'_{inner} = 100\ mm - 2t$

$b'_{inner} = 50\ mm - 2t$

$$\Rightarrow I'_{inner} = \frac{(50mm - 2t)(100mm - 2t)^3}{12}$$

Again, $I_{box} = I_{outer} - I_{inner}$

So the applicable equation is:

$$\boxed{\frac{1}{12}\left[5000\ mm^4 - (50mm - 2t)(100mm - 2t)^3\right] = 5.772\times10^{7}\ mm^4}$$

$\rightarrow$ solve for t

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PROBLEM #1M (continued)

(e) How do the answers to parts (a), (b), (c), and (d) change if aluminum (E = 67 GPa ν = 0.3) is used rather than steel? Explain carefully.

The answers to parts (a), (b) and (c) depend only on static equilibrium, so these results $\boxed{\text{do not change}}$ with a different material.

The answer to part (d) depend on static equilibrium [M(x)], the cross-section (I) and the material (E). However in considering a ratio of deflections for the case of the same material, the contribution of the material via modulus cancels out and only the ratio of I remains, thus the answer again $\boxed{\text{does not change}}$

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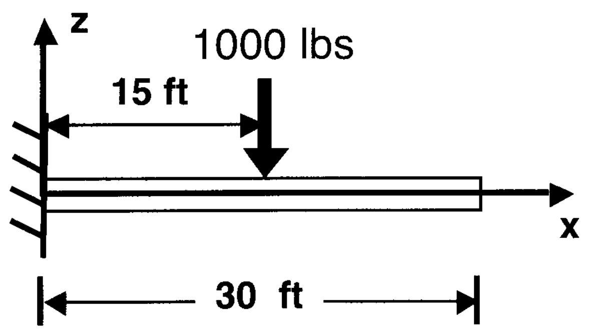
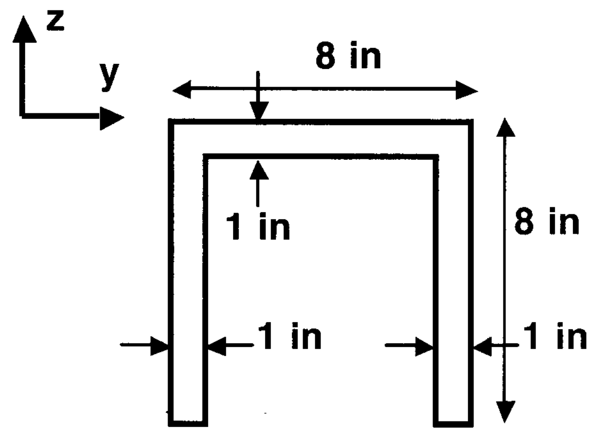
PROBLEM #2M (50%)

A statically determinate beam has a C-shape cross-section with the dimensions and orientation as in the accompanying figure. The beam is cantilevered and is loaded by a mid-span downward bending load of 1000 pounds. The beam is 30 feet long and is made of titanium with a modulus of 15 Msi.



Cross-Section



(a) Determine the ratio of the maximum tensile axial stress to the maximum compressive axial stress (i.e. σ_{xx}). Indicate the location of these maximum stresses in the cross-sectional plane.

The applicable equation is:

$$\sigma_{xx} = -\frac{Mz}{I}$$

At any location, x, along the beam, the ratio is:

$$\frac{\sigma_{xx\,max+}}{\sigma_{xx\,max-}} = \frac{-Mz_{top}/I}{-Mz_{bottom}/I} = \frac{z_{top}}{z_{bottom}}$$

We know the beam bends down due to the configuration (downward load of cantilevered beam), so top is in tension and bottom in compression.

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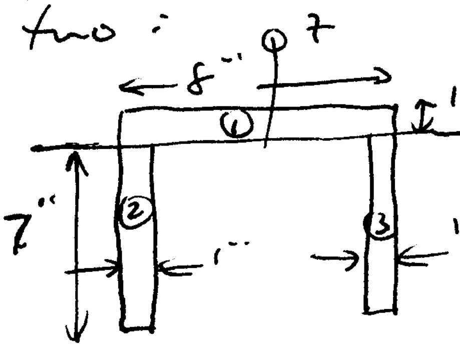
PROBLEM #2M (continued)

We are thus left with finding the location of the centroid to get z_{top} and z_{bottom}.

Thus use:

$$z_{centroid} = \frac{\iint z\,dA}{\iint dA}$$

Any point can be chosen as the initial reference. I choose the line along the bottom of the upper piece and take that for one piece and then the other two:



$A_{①} = 8\,in^2 \quad A_{②} = 7\,in^2 \quad A_{③} = 7\,in^2$

$$①: \iint z\,dA = 8'' \int_0^{1''} z\,dz = 8'' \left.\frac{z^2}{2}\right]_0^{1''} = 4\,in^3$$

$$② \text{ and } ③: \iint z\,dA = 1'' \int_{-7''}^{0} z\,dz = 1'' \left.\frac{z^2}{2}\right]_{-7''}^{0} = -24.5\,in^3$$

So

$$z_{centroid} = \frac{\sum_{①,②,③} \left(\iint z\,dA\right)}{\sum_{①,②,③} \iint dA} = \frac{-45\,in^3}{22\,in^2} = -\frac{45}{22}\,in$$

So:

$$z_{top} = 1'' - \left(-\frac{45}{22}\,in\right) = \frac{67}{22}\,in$$

$$z_{bottom} = -7'' - \left(-\frac{45}{22}\,in\right) = -\frac{109}{22}\,in$$

Thus: $\boxed{\left|\frac{\sigma_{xx\,max+}}{\sigma_{xx\,max-}}\right| = \frac{67/22\,in}{109/22\,in} = \frac{67}{109}}$ $(= 0.615)$

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PROBLEM #2M (continued)

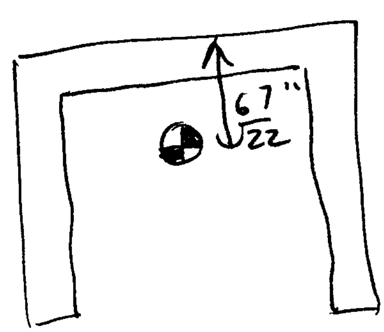
(b) Determine the location of the maximum value of the transverse shear stress, σ_{xz}, in the cross-sectional plane.

The applicable equation is:

$$\sigma_{xz} = -\frac{SQ}{Ib}$$

The maximum value of Q occurs at the centroid.
The total b (1" + 1") is a minimum there so Q/b will be at a maximum at the centroid.

So maximum magnitude value of σ_{xz} is $\boxed{\text{at the centroid}}$ – $\frac{67}{22}$ in from the top of the section



PROBLEM #2M (continued)

(c) In addition to the bending load in the z-direction, the structural configuration is subjected to a tip tensile load in the x-direction of the same magnitude. Clearly explain if and how this affects the location of the maximum tensile axial stress, compressive axial stress, and transverse shear stress.

For a pure rod case, the axial stress is:

$$\sigma_{xx_{rod}} = P/A$$

The solution for the bending case

$$\sigma_{xx_{bending}} = -\frac{Mz}{I}$$

can be added with that for the rod to get the total stress:

$$\sigma_{xx_{total}} = \sigma_{xx_{bending}} + \sigma_{xx_{rod}}$$

Thus the <u>location</u> of the maximum axial stresses (tensile and compressive)

$\boxed{\text{do not change}}$

The only possibility is that $\sigma_{xx_{rod}}$ is so large that there is no longer any compressive σ_{xx}.

For σ_{xz}, the rod loading has not affect, so by adding the two, the stress is the same and the location of the maximum transverse shear stress $\boxed{\text{does not change}}$